Effective geometry for light traveling in material media

V. A. De Lorenci*
Instituto de Ciências–Escola Federal de Engenharia de Itajubá, Avenida BPS, 1303 Pinheirinho, 37500-903 Itajubá, MG, Brazil
(Received 4 September 2001; published 25 January 2002)

Working with electrodynamics in the geometrical optics approximation, we derive the expression representing the "effective geometry" seen by electromagnetic waves propagating in media whose physical properties depend on an external electric field. Some previous results are generalized and some special cases are recovered.

DOI: 10.1103/PhysRevE.65.026612 PACS number(s): 42.25.−p, 02.40.Ky

Material media are seen by electromagnetic waves as a curved space-time described by an effective metric $g_{\mu\nu}$, which consists in a modification of the Minkowski background metric $\eta_{\mu\nu}$. This fact allows one to make an analogy between wave propagation in nontrivial media and gravitational phenomena. One work in this direction was performed by Gordon [1], in which a particular moving isotropic dielectric medium appears to light as a gravitational field demanding light to follow paths of the geometry

$$g^{\mu\nu}=\eta^{\mu\nu}-(1-\epsilon\mu)V^{\mu}V^{\nu}, \tag{1}$$

where V^{μ} represents the velocity four-vector of an observer comoving with the medium observer. Analog models for general relativity have extensively been examined in the last years, not only with respect to electrodynamics but also in the context of acoustic perturbations (for a good review on this topic, see Ref. [2], and references therein). Recently, some authors dedicated their efforts in the study of light propagating in material media [3–8]. In such a situation, Maxwell's equations must be supplemented by constitutive laws that relate the electromagnetic excitations $\vec{D}$, $\vec{B}$ and the field strengths $\vec{E}$, $\vec{H}$ by means of quantities characterizing each medium where the waves are propagating.

In this paper, assuming the media to be generally anisotropic and working with Maxwell's theory in the geometrical optics approximation, we present the effective geometry determining the possible paths of light in terms of quantities associated to the properties of the medium. We examined some special cases where the effective metric relates known situations, and finally, make some comments on the application and importance of the effective geometry interpretation. We work in Minkowski space-time, employing a Cartesian coordinate system. The background metric will be represented by $\eta_{\mu\nu}$, which is defined by diag $(+1, -1, -1, -1)$. Units are such that $c=1$.

$F_{\mu\nu}$ and $P_{\mu\nu}$ are the tensors representing the total electromagnetic field, which are expressed in terms of the strengths and the excitations of the electric and magnetic fields as

$$F_{\mu\nu}=V_{\mu}E_{\nu}-V_{\nu}E_{\mu}-\eta_{\mu\nu}{}^{\alpha\beta}V_{\alpha}B_{\beta}, \tag{2}$$

$$P_{\mu\nu}=V_{\mu}D_{\nu}-V_{\nu}D_{\mu}-\eta_{\mu\nu}{}^{\alpha\beta}V_{\alpha}H_{\beta}, \tag{3}$$

where the Levi-Civita tensor introduced is defined such that $\eta^{0123}=+1$.

In general, the properties of the media are determined by the tensors $\varepsilon_{\alpha\beta}$ and $\mu_{\alpha\beta}$, which relate the electromagnetic excitation and the field strength by the generalized constitutive laws,

$$D_{\alpha}=\varepsilon_{\alpha}{}^{\beta}(E_{\mu},H_{\mu})E_{\beta}, \tag{4}$$

$$B_{\alpha}=\mu_{\alpha}{}^{\beta}(E_{\mu},H_{\mu})H_{\beta}. \tag{5}$$

In the absence of sources, Maxwell's theory can be summarized by the equations

$$V^{\mu}D^{\nu}{}_{,\nu}-V^{\nu}D^{\mu}{}_{,\nu}-\eta^{\mu\nu\alpha\beta}V_{\alpha}H_{\beta,\nu}=0, \tag{6}$$

$$V^{\mu}B^{\nu}{}_{,\nu}-V^{\nu}B^{\mu}{}_{,\nu}+\eta^{\mu\nu\alpha\beta}V_{\alpha}H_{\beta,\nu}=0, \tag{7}$$

which are equivalent to $P^{\mu\nu}{}_{,\nu}=0$ and $\overset{*}{F}{}^{\mu\nu}{}_{,\nu}=0$, respectively. Additionally, the electromagnetic excitation is related to the field strength by means of the constitutive relations (4) and (5), whose derivatives with respect to the coordinates can be presented as

$$D_{\alpha,\tau}=\varepsilon_{\alpha}{}^{\beta}E_{\beta,\tau}+\frac{\partial\varepsilon_{\alpha}{}^{\beta}}{\partial E_{\mu}}E_{\beta}E_{\mu,\tau}+\frac{\partial\varepsilon_{\alpha}{}^{\beta}}{\partial H_{\mu}}E_{\beta}H_{\mu,\tau}, \tag{8}$$

$$B_{\alpha,\tau}=\mu_{\alpha}{}^{\beta}H_{\beta,\tau}+\frac{\partial\mu_{\alpha}{}^{\beta}}{\partial E_{\mu}}H_{\beta}E_{\mu,\tau}+\frac{\partial\mu_{\alpha}{}^{\beta}}{\partial H_{\mu}}H_{\beta}H_{\mu,\tau}. \tag{9}$$

In order to determine the propagation of waves associated to the electromagnetic field, we will consider the method of field discontinuities [9,10]. We define a surface of discontinuity Σ by $z(x^{\mu})=0$. Whenever Σ is a global surface, it divides the space-time in two distinct regions U^{-} for $z<0$, and U^{+} for $z>0$. The discontinuity of an arbitrary function $f(x^{\alpha})$ on Σ is given by

$$[f(x^{\alpha})]_{\Sigma}:=\lim_{\{P^{\pm}\}\to P}[f(P^{+})-f(P^{-})] \tag{10}$$

with P^{+}, P^{-}, and P belonging to U^{+}, U^{-}, and Σ, respectively. The electric and magnetic fields are continuous when crossing the surface Σ. However, its derivatives behave as

*Electronic address: lorenci@efei.br

$[E_{\mu,\nu}]_\Sigma = e_\mu K_\nu$ and $[H_{\mu,\nu}]_\Sigma = h_\mu K_\nu$, where e_μ and h_μ are related with the polarization of the electromagnetic waves and K_λ is the propagation vector. Applying these conditions to the field equations (6) and (7), we obtain the following set of equations governing the wave propagation:

$$\varepsilon^{\alpha\beta}K_\alpha e_\beta + \frac{\partial\varepsilon^{\alpha\beta}}{\partial E_\mu}E_\beta K_\alpha e_\mu + \frac{\partial\varepsilon^{\alpha\beta}}{\partial H_\mu}E_\beta K_\alpha h_\mu = 0, \quad (11)$$

$$\mu^{\alpha\beta}K_\alpha h_\beta + \frac{\partial\mu^{\alpha\beta}}{\partial E_\mu}H_\beta K_\alpha e_\mu + \frac{\partial\mu^{\alpha\beta}}{\partial H_\mu}H_\beta K_\alpha h_\mu = 0, \quad (12)$$

$$\left(\varepsilon^{\mu\beta}{}_{e_\beta} + \frac{\partial\varepsilon^{\mu\beta}}{\partial E_\alpha}E_\beta e_\alpha + \frac{\partial\varepsilon^{\mu\beta}}{\partial H_\alpha}E_\beta h_\alpha\right)(KV) + \eta^{\mu\nu\alpha\beta}K_\nu V_\alpha h_\beta = 0, \quad (13)$$

$$\left(\mu^{\mu\beta}h_\beta + \frac{\partial\mu^{\mu\beta}}{\partial E_\alpha}H_\beta e_\alpha + \frac{\partial\mu^{\mu\beta}}{\partial H_\alpha}H_\beta h_\alpha\right)(KV) - \eta^{\mu\nu\alpha\beta}K_\nu V_\alpha e_\beta = 0, \quad (14)$$

where we have defined $(KV) \coloneqq K^\mu V_\mu$. [The first two equations above came from the zeroth component of Eqs. (6) and (7), and correspond to the generalized Gauss laws for electric and magnetic fields.]

For those cases where $\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta}(E_\mu, H_\mu)$ and $\mu_{\alpha\beta} = \mu(\eta_{\alpha\beta} - V_\alpha V_\beta)$, the above system of equations governing the wave propagation reduces to the scalar equation

$$\varepsilon^{\ \nu\alpha}K_\nu e_\alpha + \frac{\partial\varepsilon^{\nu\beta}}{\partial E_\alpha}K_\nu E_\beta e_\alpha + \frac{1}{\mu(KV)}\frac{\partial\varepsilon^{\nu\beta}}{\partial H_\mu}\eta_{\mu\tau\sigma}{}^{\alpha}E_\beta K^\tau K_\nu V^\sigma e_\alpha = 0, \quad (15)$$

which follows from Eqs. (11) and (14), and to the eigenvector problem

$$\left[\mu(KV)^2\left(\varepsilon^{\mu\beta} + \frac{\partial\varepsilon^{\mu\alpha}}{\partial E_\beta}E_\alpha\right) + (KV)\frac{\partial\varepsilon^{\mu\rho}}{\partial H_\alpha}\eta_\alpha{}^{\tau\sigma\beta}E_\rho K_\tau V_\sigma - K^\mu K^\beta + (KV)V^\mu K^\beta + K^2\eta^{\mu\beta} - (KV)^2\eta^{\mu\beta}\right]e_\beta = 0, \quad (16)$$

which results from Eqs. (13) and (14). The Fresnel equation represents nontrivial solutions of the Eq. (16) and is given by the determinant of the term multiplied by e_β.

Now, let us analyze anisotropic media whose physical properties are influenced by an external electric field as

$$\varepsilon^{\mu\beta} = \varepsilon(E)(\eta^{\mu\beta} - V^\mu V^\beta) - \alpha(E)E^\mu E^\beta, \quad (17)$$

where we denoted $E^\alpha E_\alpha = -E^2$. Indeed, such anisotropy turns out to be a reaction of the material medium to the influence of the external electric field. Each particular medium is then characterized by the parameters α and ϵ. Considering Eq. (17), we obtain from Eq. (15)

$$e^\beta K_\beta = \frac{1}{E(\epsilon + \alpha E^2)}\frac{\partial}{\partial E}(\epsilon + \alpha E^2)E^\alpha E^\beta e_\alpha K_\beta. \quad (18)$$

Taking the product of E_μ with Eq. (16) and introducing Eq. (18), it results in the light cone condition

$$K^2 = \left(1 - \mu\frac{\partial\Omega}{\partial E}\right)(KV)^2 + \frac{1}{\Omega}\frac{\partial}{\partial E}\left(\frac{\Omega}{E}\right)E^\mu E^\nu K_\mu K_\nu, \quad (19)$$

where we introduced the quantity $\Omega = E(\epsilon + \alpha E^2)$. The condition (19) on wave propagation can be presented in a more appealing form as

$$g^{\mu\nu}K_\mu K_\nu = 0, \quad (20)$$

where

$$g^{\mu\nu} = \eta^{\mu\nu} - \left(1 - \mu\frac{\partial\Omega}{\partial E}\right)V^\mu V^\nu - \frac{1}{\Omega}\frac{\partial}{\partial E}\left(\frac{\Omega}{E}\right)E^\mu E^\nu. \quad (21)$$

Equation (20) states that the discontinuities of the electromagnetic field propagate along null geodesics of an effective geometry, which for the media defined by Eq. (17) is determined by Eq. (21). The associated phase velocity yields

$$v^2 = \frac{1}{\mu\,\partial\Omega/\partial E}\left[1 + \frac{1}{\Omega}\frac{\partial}{\partial E}\left(\frac{\Omega}{E}\right)(\vec{E}\cdot\hat{k})^2\right], \quad (22)$$

where $\hat{k}$ is a spacelike unit vector in the $\vec{K}$ direction. In the derivation of the expression (22), we have assumed the particular choice of the velocity $V_\mu = \delta^0_\mu$. The refraction index is given by $n = 1/v$.

To study birefringence phenomena, we have to solve the eigenvector problem (16). In fact, by choosing an appropriated basis [7] to expand the polarization four-vector $e_\mu = aE_\mu + bH_\mu + cK_\mu + dV_\mu$, we obtain that there will be two different light rays propagating in the medium. An ordinary ray with velocity

$$v_o^2 = \frac{1}{\mu(\epsilon + \alpha E^2)}\quad, \quad (23)$$

and another one, the extraordinary ray, with velocity given by Eq. (22). The velocity of the extraordinary ray takes the same value of the one corresponding to the ordinary ray in the case where $\vec{E}\cdot\hat{k} = |\vec{E}|$. Both velocities will be the same in the case where $\epsilon + \alpha E^2 = \text{const}$. Thus, birefringence of the electromagnetic waves occurs whenever $\partial(\Omega/E)/\partial E \neq 0$.

Let us summarize some cases contained in the effective metric (21):

(a) For $\alpha = 0$ and ϵ constant, results the Gordon [1,5] effective metric (1).

(b) For $\alpha = 0$ and $\epsilon = \epsilon(E)$, we obtain the isotropic case [6,7] with propagation determined by

$$g^{\mu\nu} = \eta^{\mu\nu} - \left[1 - \mu\frac{\partial(\epsilon E)}{\partial E}\right]V^\mu V^\nu - \frac{1}{\epsilon E}\frac{\partial\epsilon}{\partial E}E^\mu E^\nu. \quad (24)$$

(c) For α and ϵ constants, we obtain the effective geometry

$$g^{\mu\nu}=\eta^{\mu\nu}-[1-\mu(\epsilon+3\alpha E^2)]V^\mu V^\nu+\frac{2\alpha}{(\epsilon+\alpha E^2)}E^\mu E^\nu, \tag{25}$$

which determines the Kerr electro-optic effect [7,12].

In this paper, we derived the light cone conditions for light propagating inside generally anisotropic material media whose physical properties depend on an external electric field. The wave vector appears as a null vector of an effective curved geometry representing a modification of the flat background metric. Indeed, if we require an underlying Riemannian structure for the manifold associated with the effective geometry it can be shown [11] that the integral curves of the vector K_μ are geodesics, i.e., satisfy the geodesic equation

$$g^{\mu\nu}K_{\alpha;\mu}K_\nu=0. \tag{26}$$

To apply the formalism developed here, we need to know the properties of each particular medium where the wave propagation is considered by means of the constitutive relations characterized by the tensors $\varepsilon_{\mu\nu}$ and $\mu_{\mu\nu}$.

An effective geometry can be derived for each situation and can be used in the study of the properties of light propagation. With such geometrical description, we present tools for testing kinematic aspects of gravitation in laboratory, an issue very much addressed these days [2–6]. For instance, we could ask about the possibility of formation of structures such as event horizons.

The derivation of the effective geometry associated with arbitrary media characterized by $\varepsilon_{\mu\nu}(E_\alpha,H_\alpha)$ and $\mu_{\mu\nu}(E_\alpha,H_\alpha)$ deserves further investigation.

The author is grateful to R. Klippert and E. S. Moreira, Jr. for reading the manuscript and for valuable suggestions. This work was partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) of Brazil.

[1] W. Gordon, Ann. Phys. (Leipzig) **72**, 421 (1923).
[2] C. Barceló, S. Liberati, and M. Visser, Class. Quantum Grav. **18**, 1137 (2001).
[3] Y. N. Obukhov and F. W. Hehl, Phys. Lett. B **458**, 466 (2000); M. Schonberg, Rev. Bras. Fis. **1**, 91 (1971).
[4] Y. N. Obukhov, T. Fukui, and G. Rubilar, Phys. Rev. D **62**, 044050 (2000).
[5] U. Leonhardt and P. Piwnicki, Phys. Rev. Lett. **84**, 822 (2000); M. Visser, *ibid.* **85**, 5252 (2000); **85**, 5253 (2000).
[6] M. Novello and J. M. Salim, Phys. Rev. D **63**, 083511 (2001).
[7] V. A. De Lorenci and M. A. Souza, Phys. Lett. B **512**, 417 (2001).
[8] I. Brevik and G. Halnes, gr-qc/0106045.
[9] J. Hadamard, in *Leçons sur la Propagation des Ondes et les Équations de L'Hydrodynamique* (Hermann, Paris, 1903); J. Plebanski, in *Lectures on Non-linear Electrodynamics* (Nordita, Copenhagen, 1968); G. Boillat, J. Math. Phys. **11**, 941 (1970).
[10] V. A. De Lorenci, R. Klippert, M. Novello, and J. M. Salim, Phys. Lett. B **482**, 134 (2000).
[11] M. Novello, V. A. De Lorenci, J. M. Salim, and R. Klippert, Phys. Rev. D **61**, 045001 (2000).
[12] L. Landau and E. Lifchitz, *Électrodynamique des Milieux Continus* (Mir, Moscow, 1969).